The Moutard transformation and two-dimensional multi-point delta-type potentials

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Let H be a two-dimensional Schrödinger operator $H = -\Delta + U = -4\bar{\partial}\partial + U$, where $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $x, y \in \mathbb{R}$, and let ω be a formal solution to the equation

$$H\omega = 0. \tag{1}$$

The Moutard transformation corresponds to H and ω the operator

$$\widetilde{H} = -4\bar{\partial}\partial + \widetilde{U} = -4\bar{\partial}\partial + U - 8\bar{\partial}\partial \log \omega \tag{2}$$

such that for every φ meeting the equation $H\varphi = 0$ a function θ satisfying the system

$$(\omega\theta)_z = -i\omega^2 \left(\frac{\varphi}{\omega}\right)_z, \quad (\omega\theta)_{\bar{z}} = i\omega^2 \left(\frac{\varphi}{\omega}\right)_{\bar{z}}, \tag{3}$$

satisfies $\widetilde{H}\theta = 0$. The function θ is defined modulo $\frac{1}{\omega}$ due to the integration constant in the right-hand sides of (3).

Recently the Moutard transformation which originates in the surface theory was used for constructing special types of two-dimensional potentials and blowing up solutions of the Novikov–Veselov equation [1, 2].

In difference with [1] which concerns with regular potentials in the present note we deal with multi-point delta-type potentials. We consider also the Faddeev eigenfunctions [3] of the corresponding operators H on the zero energy level. These eigenfunctions are defined by conditions

$$H\psi = 0, \quad \psi(z, \bar{z}, \lambda) = e^{\lambda z}(1 + o(1)) \quad \text{as } z \to \infty,\ \lambda \in \mathbb{C} \setminus \{0\}.$$

In addition,

$$\psi = e^{\lambda z}\left(1 + \frac{a(\lambda, \bar{\lambda})}{z} + e^{\bar{\lambda}\bar{z} - \lambda z}\frac{b(\lambda, \bar{\lambda})}{\bar{z}} + o\left(\frac{1}{|z|}\right)\right) \text{ as } z \to \infty,$$

where a, b are the Faddeev generalized "scattering" data on the zero energy level.

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Theorem 1 *A formal application of the Moutard transformation to the zero potential $U = 0$ by using a polynomial in z function $\omega = P(z) = \prod_{k=1}^{N}(z - z_k)$ leads to the milti-point delta-type potential*

$$\widetilde{U}(z) = -8\pi \sum_{k=1}^{N} \delta(z - z_k).$$

For this potential the Faddeev eigenfunctions on the zero energy level take the form

$$\psi = e^{\lambda z}\left(1 + \frac{2}{P}\sum_{k=1}^{N}\frac{(-1)^k P^{(k)}(z)}{\lambda^k}\right), \tag{4}$$

where $P^{(k)}(z) = \partial^k P(z)$. In addition, for these eigenfunctions $a = -2N/\lambda$ and $b \equiv 0$.

The proof of this theorem is based on solving system (3) with respect to $\psi = \theta$ for $\omega = P(z)$ and $\varphi = ie^{\lambda z}$, and on straightforward computations. However we need to clarify the meaning of Schrödinger operators with such potentials.

Actually in this case we consider the Moutard system (3) with $\omega = P(z)$ as the appropriate regularization of the Schrödinger equation $\widetilde{H}\theta = 0$ with the potential $\widetilde{U}$ of Theorem 1. In addition, for $N = 1$, the Schrödinger equation $(-\Delta + \widetilde{U})\psi = 0$ with $\widetilde{U}$ and ψ from Theorem 1 is formally fulfilled under the following conventions:

$$\bar{\partial}\left(e^{\lambda z}\left(\frac{1}{z}\right)^2\right) = e^{\lambda z}\frac{2}{z}\bar{\partial}\left(\frac{1}{z}\right) = \frac{2\pi e^{\lambda z}\delta(z)}{z}.$$

We remark that the functions ψ of (4) essentially differ from the Faddeev eigenfunctions found in [4, 5] for the Schrödinger operators with multi-point delta-type potentials. The reason is that in [4, 5] the operator with such a potential is replaced by its regularization going back to [6], whereas in the present note we work formally with the original potentials considering the regularization, of the equation $\widetilde{H}\theta = 0$, given by the Moutard system (3).

In addition, in view of the property $b \equiv 0$ for ψ of (4) the potentials of theorem 1 may be considered as "reflectionless" in the sense of the Faddeev generalized "scattering" data a, b. In this sense the functions ψ of (4) are similar to the Faddeev eigenfunctions found in [9] for some regular potentials.

In [1, 8] the Moutard transformation is extended to a transformation of solutions of the Novikov–Veselov equation [7]

$$U_t = \partial^3 U + \bar{\partial}^3 U + 3\partial(UV) + 3\bar{\partial}(U\bar{V}) = 0, \quad -4\bar{\partial}V = \partial U. \tag{5}$$

This equation has the Manakov form $H_t = HA + BH$ where A and B are differential operators. If U satisfies (5) and ω meets (1) and the equation

$$(\partial_t + A)\omega = 0, \tag{6}$$

then the extended Moutard transformation of U has the same form (2) and gives a new solution of (5). For the zero potential $U = V = 0$, we have $A = \partial^3 + \bar{\partial}^3$ and $\omega(z,t) = P(z,t) = \prod_{k=1}^N (z - z_k(t))$ satisfies (6) if and only if

$$\frac{\partial P}{\partial t} = \frac{\partial^3 P}{\partial z^3}.$$

The latter equation describes an algebraic dynamics of the zeroes of $P(z,t)$ (such a dynamics for another reason was considered in [1]). A formal application of the extended Moutard transformation leads to the potential

$$\widetilde{U}(z,t) = -8\pi \sum_{k=1}^N \delta(z - z_k(t))$$

which apparently may be considered as a formal solution to (5).

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